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Theoretical confirmation of Feynman's hypothesis on the creation of circular vortices in Bose–Einstein condensates: II

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Abstract

In a recent paper (Infeld and Senatorski 2003 *J. Phys.: Condens. Matter* **15** 5865) we confirmed Feynman's hypothesis on how circular vortices can be created from an oppositely polarized linear pair in a Bose–Einstein condensate. This was done by perturbing the original pair numerically, so that a circular vortex (or array of identical circular vortices) was created as a result of reconnection. These circular vortices were then checked against known theoretical relations binding velocities and radii. Agreement to a high degree of accuracy was found. Here in part II, we give examples of the creation of several *different* vortices from one linear pair. All are checked as above. We also confirm the limit of separation of the line vortices below which mutual attraction, followed by annihilation, prevents the Feynman metamorphosis. Other possible modes of behaviour are illustrated.

1. Introduction

The experimental realization of Bose–Einstein condensation in alkali metal gases has renewed interest in all aspects of these media. The formation and dynamics of vortices is at the forefront of this effort. However, much of interest can be found in classical helium 4 work. In this context, almost half a century ago, Feynman [1] postulated that two oppositely polarized line vortices in a Bose–Einstein condensate (BEC) could cross at two points and then reconnect so as to create a circular vortex. This vortex would next snap off and live a life of its own. (This was also assumed to be possible for two opposite parallel segments of one very large vortex. In fact the illustration in the original paper pictures just this situation.) This hypothesis was used in the literature but not proven until recently. It was proven in [2], henceforth referred to as part I. However, several initial steps were required, as outlined there [3–7]. We will now broaden the analysis of part I by giving examples of the creation of two or more different circular vortices when the linear antecedents cross at three or more points. We will also find that there is a limit to how close the linear vortices can be in terms of the *healing length*, before they attract each other and rapid annihilation results. No circular vortices emerge in this last mentioned scenario.

This paper can be read independently of part I.

2. Basic equations

A single-component BEC can be described by a single-particle wavefunction of N bosons of mass m that obeys the equation formulated independently by Gross and Pitaevski [8, 9] and now carries both of their names:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + W_0\psi|\psi|^2. \tag{1}$$

Here W_0 characterizes the potential between bosons, assumed positive in our treatment. Both opposite linear pair and circular vortex solutions that are stationary are known [3]. Each solution has a unique velocity perpendicular to the vortex plane. So as to give a universal curve for the dependence of this velocity on the radius for a circular vortex, we remove the coefficients in equation (1) by rescaling. To do this we introduce the transformation (μ is the chemical potential, ρ_0 the background density)

$$\psi \to \sqrt{\rho_0}\mathrm{e}^{-\mathrm{i}\mu t/\hbar}\psi, \qquad \mathbf{x} \to \frac{\hbar}{\sqrt{2\mu m}}\mathbf{x}, \qquad t \to \frac{\hbar}{2\mu}t, \qquad \mu = W_0\rho_0, \tag{2}$$

and obtain

$$2\mathrm{i}\frac{\partial\psi}{\partial t} = -\nabla^2\psi - \psi(1-|\psi|^2). \tag{3}$$

The unit of length introduced in equation (2) is known as the *healing length*. If we write $\psi = \rho^{1/2}\mathrm{e}^{\mathrm{i}S}$, then ρ and $\mathbf{v} = \nabla S$ have a fluid interpretation (e.g. the usual equation of continuity is satisfied). However, if we encircle $\psi = 0$ once, S must increase or decrease by $\pm 2n\pi$ so ψ is single valued. This implies quantization of the circulation. Here we consider $n = 1$. Although this quantization is of course absent in classical vortices, many phenomena are similar. However, reconnection in the absence of dissipation is not, as it could not occur in a classical fluid (forbidden by the Kelvin–Helmholtz theorem).

The theoretical dependence of the radius on the velocity for circular vortices with $n = 1$ can be seen in figure 1 (for theory see part I). The speed of sound $U = 1/\sqrt{2}$. This is also an upper limit for U for all vortices, linear and circular (in fact both species were seen in [3] to disappear at slightly lower values of U).

3. Creation of sundry circular vortices from linear pairs

As initial condition, we took a two-line-vortex configuration (y is cyclic so far):

$$\psi(t=0) = \frac{r_1 r_2}{\sqrt{r_1^2+b^2}\sqrt{r_2^2+b^2}}\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}, \tag{4}$$

where

$$r_1^2 = (1-2V^2)(x+a)^2 + z^2, \qquad r_2^2 = (1-2V^2)(x-a)^2 + z^2,$$
$$\tan\theta_{1,2} = \frac{z}{\sqrt{1-2V^2}(a \pm x)}.$$

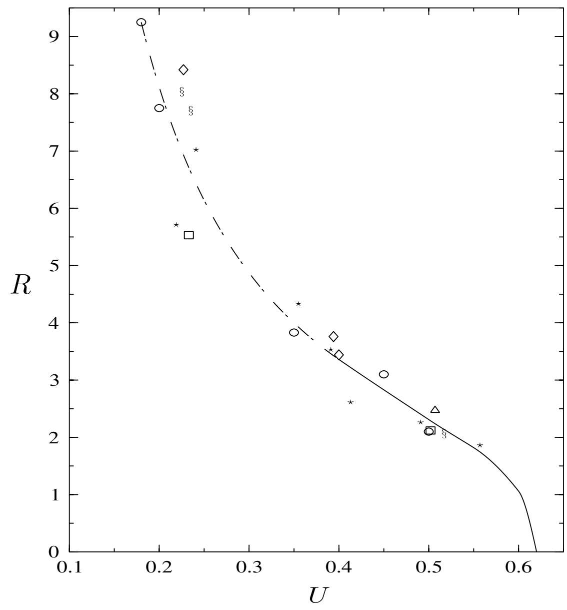

Figure 1. Radii of emerging circular vortices as a function of their velocities along z. Continuous curve: theory as follows from Jones and Roberts [3]; combined with large R theory of Roberts and Grant [3]. This combined curve appears in part I. Circles correspond to ring vortices, as found in part I; triangle, in this paper, figure 2; squares, figure 3; diamonds, figure 4; asterisks, figure 5; snakes, figure 8.

This model is discussed in part I. Once a is chosen, we fit V and b so as to approximate a solution following table 2 of the first reference of [3]. Other models, based on Padé approximants, that fit best at the mid-point between vortices, are now known [10]. Regardless of the merits of different models, we have found that the emergence of circular vortices was not sensitively dependent on the details of the initial condition. We perturbed the line vortices away from their plane (along z) with wavelengths of the perturbation a few times the separation, as can be seen in the figures. Thus we trigger the Crow instability studied theoretically for the Gross–Pitaevski equation and in the long wavelength limit [4]. (Although we have reservations about the calculation, the end result seems to be correct; see the comment following reference [4].) It was found that only a perturbation away from the plane of the vortices leads to an instability (bending towards increasing z). The nonlinear effect of this instability was reconnection as in figures 2–6.

Each emerging circular vortex oscillated initially. The amplitudes of these oscillations decreased until more or less stationary circular vortices emerged.

Figures 2–6 illustrate examples of the emergence of one to twelve circular vortices, depending on the wavelength of the initial perturbation. This initial perturbation along cyclic y was always such that at least one whole wavelength corresponded to the height of the box, though this height was varied from case to case. The dependence of radii on velocities along z for emerging circular vortices is shown in figure 1. This figure strengthens the conclusion of part I, namely that circular vortices, if created, satisfy known theoretical relations between U and R.

Figure 7 illustrates the annihilation of two line vortices when $a < a_{\mathrm{cr}} \simeq 0.85$ [11].

Figure 8 pictures the collision and fusion of two circular vortices. Points in (U, R) parameter space corresponding to situations both before and after the fusion are also shown

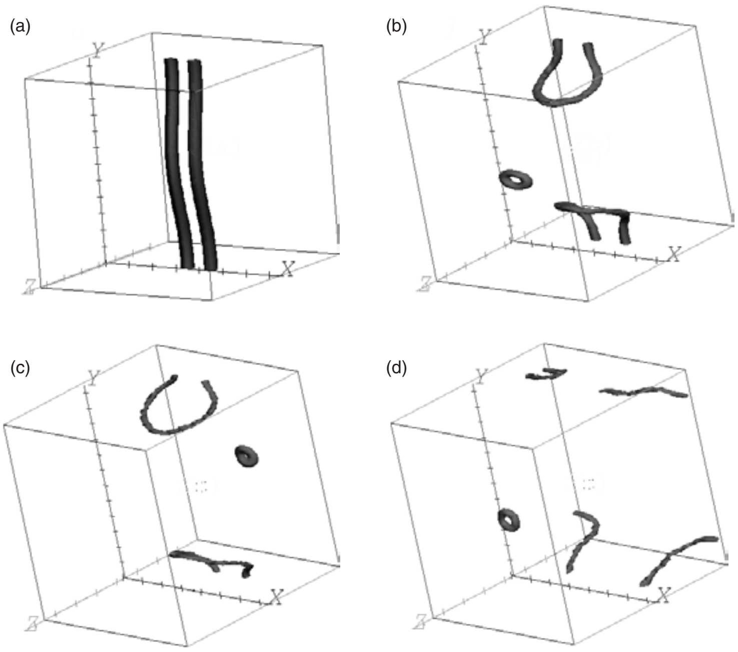

Figure 2. Emergence of a circular vortex (repeated periodically) as a result of the Crow instability of two oppositely polarized linear vortices. The instability is induced numerically. Here $a = 2.5$. The densities shown are 0.6 in first frame (for better visualization), 0.2 in the rest. Times are 0, 105, 125, 230. The distance between consecutive tick marks is five healing lengths in this and the following figures.

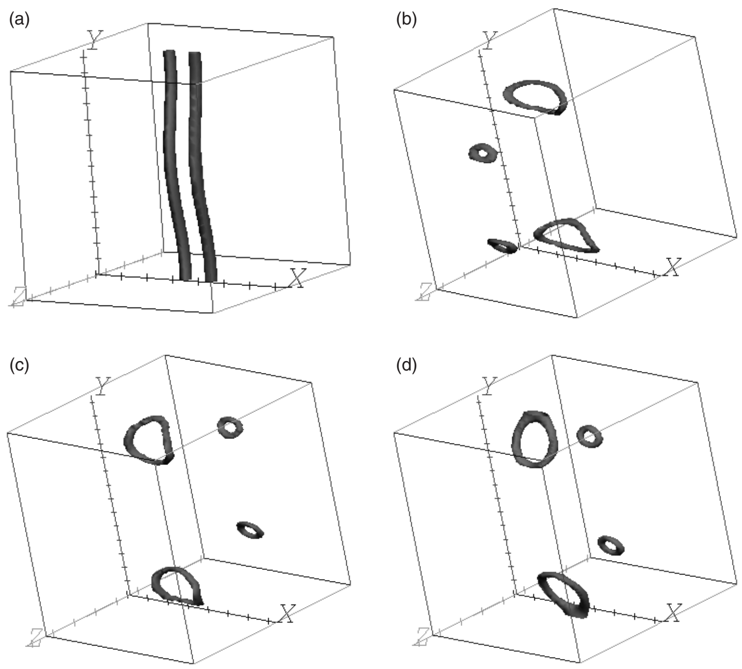

Figure 3. Emergence of two different pairs of circular vortices. The half-distance $a = 2.5$. Densities are 0.15, 0.2, 0.2, 0.2. Consecutive times are 0, 125, 150, 170.

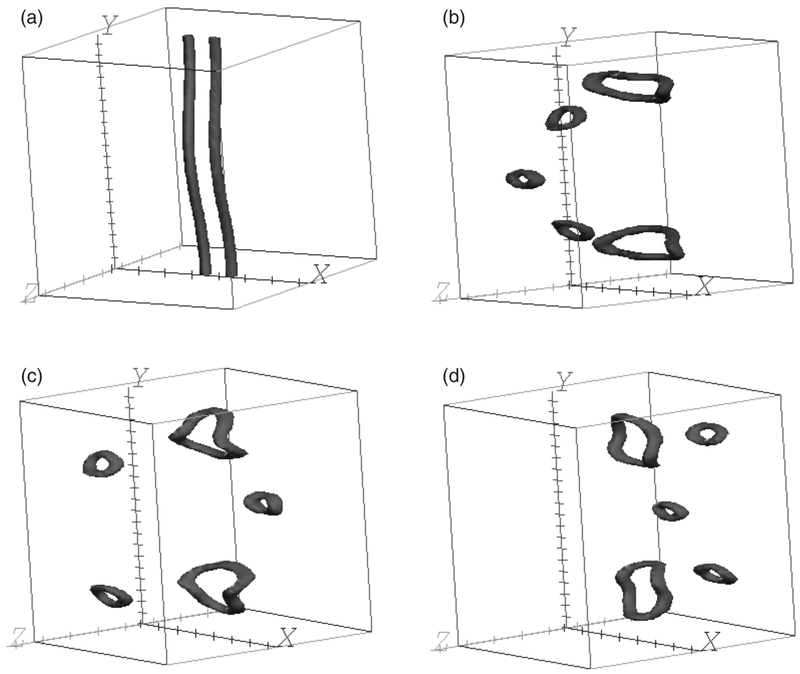

Figure 4. Emergence of five vortices (from each wavelength of the perturbation), $a = 2.5$. Densities are 0.6, 0.35, 0.35, 0.35. Times are 0, 190, 235, 255.

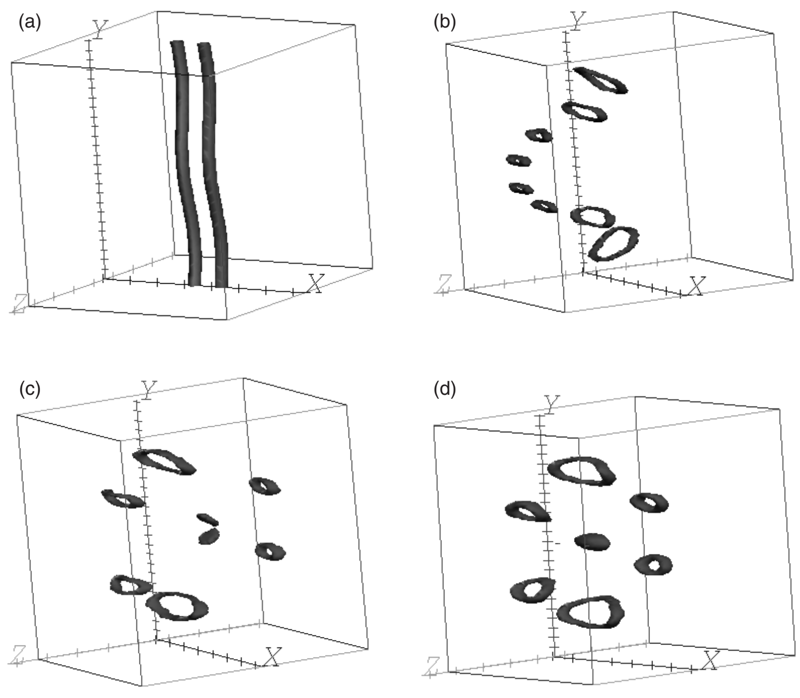

Figure 5. Emergence of several circular vortices (eight), $a = 2.5$. Densities are 0.5, 0.25, 0.33, 0.25. Times are 0, 145, 200, 230.

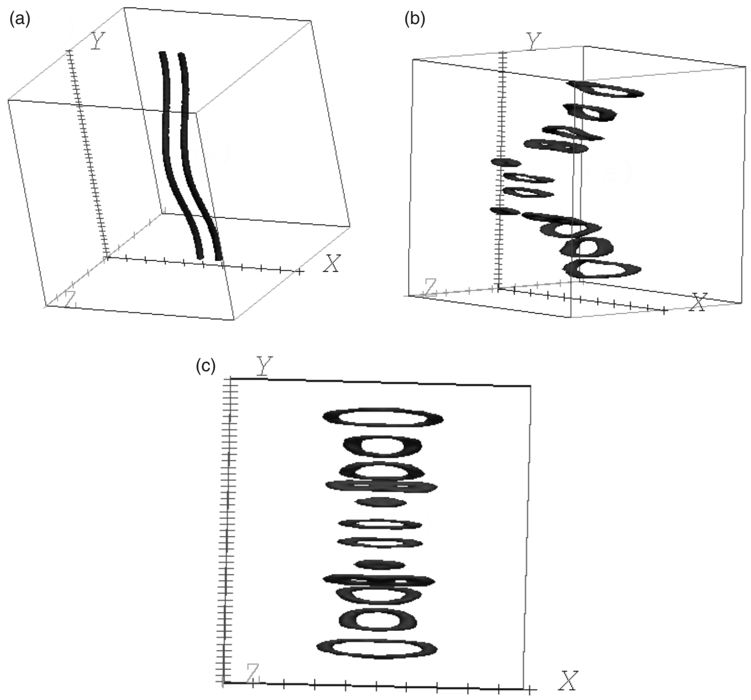

Figure 6. Emergence of twelve circular vortices from one wavelength of the perturbation, $a = 2.5$. Densities are 0.6 and 0.4. Times are 0 and 270, the latter two frames picturing different perspectives of the same configuration for clarity.

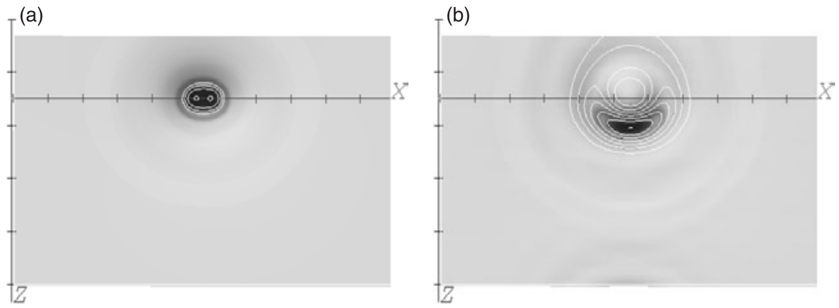

Figure 7. Mutual annihilation of two line vortices. According to the theory of [11], this happens if $a < a_{\mathrm{cr}} \simeq 0.85$. Here $a = 0.7$. Times are 0 and 5. Note the difference in timescale as compared to the other phenomena described here.

in figure 1. This type of fusion has been observed before [6], but we now see that all three participating vortices approximately satisfy the U, R dependence of known stationary solutions to equation (3) as found in the two references of [3]. This is the case once we are well away from the collision.

Figure 9 illustrates the rotation of two identically polarized line vortices, well known from classical fluid dynamics (see [12]). For them the denominator on the right-hand side of the last member of equation (4) would have a term $x \pm a$.

Numerical details of our calculations can be found in part I.

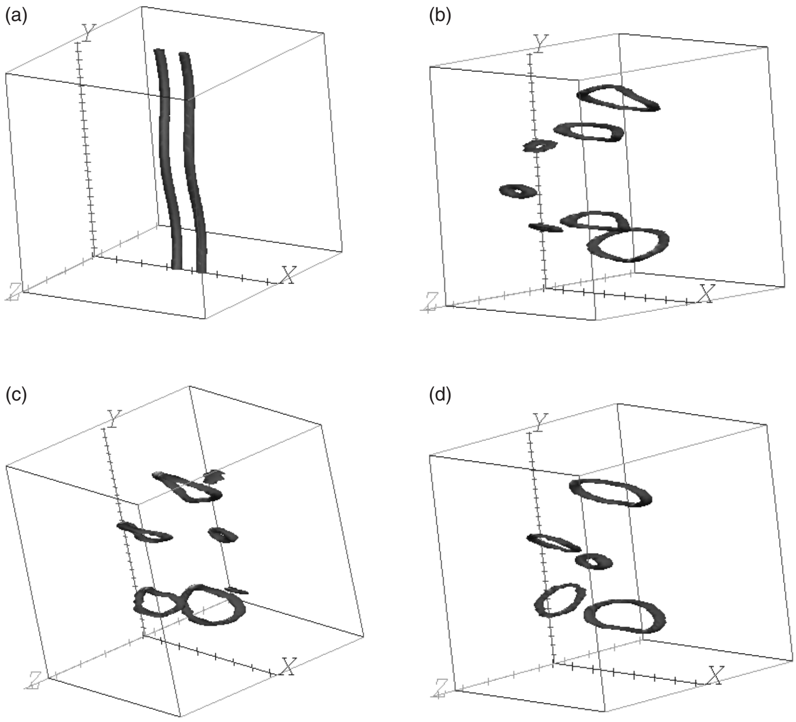

Figure 8. Collision and fusion of circular vortices (first and third from the top). The original half-distance $a = 2.5$. Densities are 0.35, 0.3, 0.27, 0.26. Times are 0, 175, 225, 275.

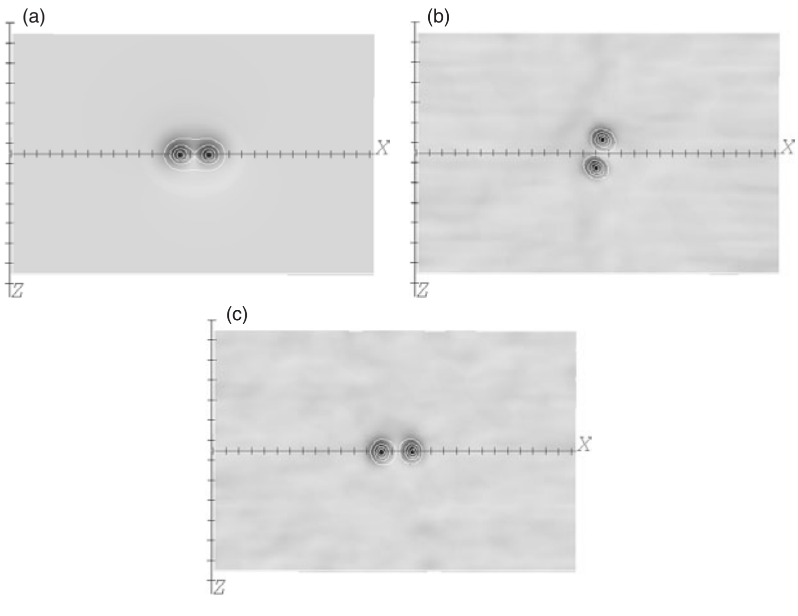

Figure 9. Rotation of two identically polarized line vortices around their mid-point, just as in a classical fluid. Note that $a \approx 3.73$ throughout. The period of the rotation is $T = 190$.

4. Summary

In part I we demonstrated how two oppositely polarized line vortices can cross at two points and next produce a robust, circular vortex. It was seen to satisfy known relations between the velocity and radius. Here in part II, we have broadened this proof. Many different circular vortices were produced from several crossings of one linear pair. All emerging vortices were seen to satisfy the above relations required for robustness, with a good degree of accuracy. The fusion of two vortices to produce a third was followed. Even then, all three were seen to basically satisfy these constraints when well separated.

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Roberts P H and Grant J 1971 *J. Phys. A: Math. Gen.* **4** 55
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